International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Stochastic Cocycles over $\mathbb{Z}_2$-Graded *-Algebras

V. R. Struleckaja[1]

Received March 28, 1995

The correspondence between fermionic evolution and stochastic cocycles over $\mathbb{Z}_2$-graded von Neumann *-algebras is proved to be parallel to the one in the boson case. To show this, the technique of unification of bosonic and fermionic quantum stochastic calculus is applied.

INTRODUCTION

It turns out that it is quite natural to characterize quantum stochastic evolution in terms of stochastic cocycles (Hudson and Lindsay, 1987; Journé, 1987; Accardi *et al.*, 1987; Fagnola, 1991). This has proved to be a very powerful tool, extending to the case when the coefficients of the corresponding evolution equation are unbounded (Accardi and Mohari, 1994). All the results mentioned have to do with bosonic stochastic evolution.

We aim here to give an outline of how this machinery can be generalized to the case of a $\mathbb{Z}_2$-graded system algebra when the evolution is driven by an equation with bounded coefficients and by fermionic noise. The main result establishes that the classes of stochastic evolutions with bounded coefficients and of cocycles with uniformly continuous reduced semigroup are coextensive. To prove this, we use the technique of unification of fermionic and bosonic quantum stochastic calculus (Hudson and Parthasarathy, 1986). Notice that this result is by no means an immediate corollary of the bosonic one.

In Section 1 we recall the useful definitions and properties of the objects with which we are concerned and introduce our notations. In Section 2 we

[1] Department of Mathematics, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom.

0020-7748/95/0800-1727$07.50/0

introduce the basic martingales of fermionic quantum stochastic calculus via bosonic ones, define the fermionic cocycle, and prove our main result. The converse statement is formulated in the last section.

1. PRELIMINARIES AND BASICS OF QUANTUM STOCHASTIC CALCULUS

Let h be a complex Hilbert space, and let $\Gamma(h)$ denote the boson Fock space over h. For each element $f \in h$ let $\psi(f)$ denote the coherent (exponential) vector $(1, f, \ldots, 1/(n!)^{1/2} \otimes_n f, \ldots)$, where $\otimes_n f$ denotes the symmetric tensor product of n copies of f. The linear span of $\{\psi(f): f \in h\}$ is total in $\Gamma(h)$. We denote it by $\mathscr{E}$. Let $h = h_t \oplus h^t$ be the direct sum decomposition of the Hilbert space h ($= L^2(\mathbb{R}^1_+) = L^2[0, t] \oplus L^2[t, \infty)$ here). We identify $\Gamma(h) = \Gamma(h_t) \otimes \Gamma(h^t)$, so that the exponential domain $\mathscr{E}$ factorizes as an algebraic tensor product $\mathscr{E} = \mathscr{E}_t \underline{\otimes} \mathscr{E}^t$. Let Γ_I denote the Fock space over $h = L^2(I)$, where I may denote $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, $[a, b] \subseteq \mathbb{R}^1_+$. We write Γ_+ for $\Gamma(L^2(\mathbb{R}^1_+))$ and Γ for $\Gamma(L^2(\mathbb{R}^1))$.

Given a contraction S on h, the second quantized contraction $\Gamma(S)$ is defined by the action $\Gamma(S)\psi(f) = \psi(Sf)$.

We denote by $\mathcal{M}_I$ the $\mathbb{Z}_2$-graded von Neumann subalgebra of a $\mathbb{Z}_2$-graded algebra $\mathcal{M} := B(\Gamma_+)$ generated by the fermion field operators $\{b^\#(f)\}$, with $f \in h$ and $\operatorname{supp} f \subseteq I \subset \mathbb{R}^1_+$, i.e., $\mathcal{M}_I := \{b^\#(f): \operatorname{supp} f \subset I\}''$. Here $b^\#(f)$ denotes annihilation $b(f)$ or the creation field operator $b^*(f)$.

A $\mathbb{Z}_2$-graded $*$-algebra $\mathcal{M}$ is a pair $(\mathcal{M}, \gamma)$ comprising of a $*$-algebra $\mathcal{M}$ and a $*$-automorphism γ of $\mathcal{M}$, s.t. $\gamma^2 = id$. We write $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, where $\mathcal{M}^\pm$ are the fixed point spaces of γ and $-\gamma$, respectively. One obtains such a γ by conjugation by a self-adjoint unitary (parity) operator R, which acts as follows: $RX = XR$, $X \in \mathcal{M}_I^+$, and $RX = -XR$, $X \in \mathcal{M}_I^-$, so that $X^\gamma = RXR$ (Hudson, 1993). We shall use the parity operator $R(t)\psi(f) = \psi(-f\chi_{[0,t]} + f\chi_{(t,\infty)})$ to $\mathbb{Z}_2$-grade $\mathcal{M}_{[0,t]}$ (Hudson and Shepperson, 1992). It is evident that this leaves invariant each of the subalgebras $\mathcal{M}_I = \mathcal{M}_I^+ \oplus \mathcal{M}_I^-$.

We denote by $\hat{\otimes}$ the $\mathbb{Z}_2$-graded tensor product [studied first by Chevalley (1955); Davies (1971) studied a skew tensor product of von Neumann algebras in the framework of involutory automorphisms of W^*-algebras; in Hudson (1993) and Hudson and Struleckaja (1995) the $\mathbb{Z}_2$-graded tensor product of the $*$-algebras appeared in the studies of fermionic stochastic flow evolutions]. We observe that if $I = [s, t]$ for $I \subset [0, t]$,

$$\mathcal{M}_I \subseteq I_{\Gamma_{[0,s]}} \hat{\otimes} \{B(\Gamma_I)^+ \oplus B(\Gamma_I)^-\} \hat{\otimes} I_{\Gamma_{(t,\infty)}}$$

consists of $\mathbb{Z}_2$-graded ampliation, where $B(\Gamma_I)^\pm$ denotes the even (+) and odd (−) bounded operators on the corresponding Fock space Γ_I.

Given an initial Hilbert space h_0 carrying the initial (system) algebra $\mathcal{M}_0 = \mathcal{M}_0^+ \oplus \mathcal{M}_0^-$ obtained by conjugation by the initial parity operator R_0, we define the $\mathbb{Z}_2$-graded von Neumann algebra $\mathcal{N}_I := \{\mathcal{M}_0 \hat{\otimes} \mathcal{M}_I\colon I \subseteq R^1_+\}''$. We equip the algebra $\mathcal{N}_I$ with the product $\mathbb{Z}_2$-graded structure (Chevalley, 1955; Hudson and Struleckaja, 1995).

The $\mathcal{N}_I$, $I = [s, t]$, $0 \leq s < t < \infty$, form a double filtration of $B(h_0 \otimes \Gamma_+)$ in the sense that $\mathcal{N}^{\pm}_{[s,t]} \subset \mathcal{N}^{\pm}_{[s',t']}$ whenever $0 < s' \leq s \leq t \leq t' < \infty$ and

$$\bigcup_{I \subseteq \mathbb{R}_+} \mathcal{N}_I = B(h_0 \otimes \Gamma_+)$$

There exists the vacuum-state conditional expectation $\mathbb{E}_I$ onto each $\mathcal{N}_I$. It is characterized by the property

$$\begin{aligned}&\mathbb{E}_I[T \hat{\otimes} b^{\#_n}(f_n) \cdots b^{\#_1}(f_1)]\\ &\quad = \langle \psi(0), b^{\#_n}(f_n \chi_{I^c}) \cdots b^{\#_1}(f_1 \chi_{I^c}) \psi(0) \rangle T \hat{\otimes} b^{\#_n}(f_n \chi_I) \cdots b^{\#_1}(f_1 \chi_I)\end{aligned}$$

where $T \in \mathcal{M}_0$, $b^{\#_n}(f_n) \cdots b^{\#_1}(f_1)$ is a polynomial in the fermion field creation and annihilation operators, $\forall i = \overline{1, n}, f_i \in L^2(\mathbb{R}^1_+)$, and I^c denote the complement of I in $\mathbb{R}^1_+$. Here $\mathbb{E}_I$ satisfies the projective property $\mathbb{E}_{I_1} \circ \mathbb{E}_{I_2} = \mathbb{E}_{I_1}$, if $I_1 \subseteq I_2 \subset \mathbb{R}^1_+$.

The bounded process $X = (X(t)\colon t \in \mathbb{R}^1_+)$, which concerns us, defined together with its adjoint $X^\dagger$ on a common domain in $h_0 \underline{\otimes} \mathscr{E}$ (algebraic tensor product) is *adapted* in the sense that $X(t) = X_t \otimes 1^t$ $[X^\dagger(t) = X_t^\dagger \otimes 1^t]$.

An adapted process X is a *martingale* if it satisfies the identity

$$\mathbb{E}_{0,s}[X(t)] = X(s), \qquad \forall s < t$$

The martingale is *regular* (Parthasarathy and Sinha, 1986) if there is a Radon measure μ on $I \subset \mathbb{R}^1_+$ such that for $s < t$, $u \in \Gamma_{0,s} \hat{\otimes} \psi(0)_{\Gamma_{s,\infty}}$,

$$[\mathbb{E}_{0,s}\|[(X(t) - X(s))]u\|^2 + \mathbb{E}_{0,s}\|[(X^*(t) - X^*(s))]u\|^2] \leq \mu([s, t])\|u\|^2$$

Then Parthasarathy and Sinha (1986) it can be represented as a stochastic integral against the three fundamental martingales of bosonic quantum stochastic calculus (see next section). We observe that generalization of this property to the case of $\mathbb{Z}_2$-graded von Neumann algebras $\mathcal{N}_I$ is straightforward (Struleckaja, 1994).

2. QUANTUM STOCHASTIC FERMIONIC COCYCLES

Let S be the right shift on $L^2(\mathbb{R}^1_+)$,

$$S_t(f)(x) = \begin{cases} f(x - t), & x \geq t \\ 0, & \text{otherwise} \end{cases}$$

The operator S_t is a contraction. Moreover, the second quantization $\Gamma(S_t)$ as well as the operator S_t itself are isometric. Also $\Gamma(S_s)\Gamma(S_t) = \Gamma(S_{s+t})$. The $\Gamma(S_t)$ is an even operator; we identify it with its ampliation to the initial space.

We introduce three basic martingales of bosonic quantum stochastic calculus and thereafter will pass onto their fermionic counterparts by applying the arguments of unification theory (Hudson and Parthasarathy, 1986). These are the gauge K^1 and the boson creation K^2 and annihilation K^3 processes defined by their action on $u \otimes \psi(f)$, $u \in h_0$, $\psi(f) \in \mathscr{E}, f \in L^2(\mathbb{R}^1_+)$ by

$$(K^1(t)u \otimes \psi(f)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} u \otimes \psi\left(\exp \int \epsilon\chi_{[0,t]}f\right)$$

$$(K^2_B(t)u \otimes \psi(f)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} u \otimes \psi(f + \epsilon\chi_{[0,t]})$$

$$(K^3_B(t)u \otimes \psi(f)) = \int_0^t f(s)\, ds\, u \otimes \psi(f) \tag{2.0}$$

K^1 is symmetric and K^2_B and K^3_B are mutually adjoint on $h_0 \underline{\otimes} \mathscr{E}$.

According to Cockroft and Hudson (1978), the usual bosonic creation and annihilation field operators can be expressed as stochastic integrals $a(f) = \int \bar{f}\, dK^3_B$, $a^\dagger(f) = \int f\, dK^2_B$, in the sense that $a(f\chi_{[0,t]}) = \int_0^t \bar{f}\, dK^3_B$ and $a^\dagger(f\chi_{[0,t]}) = \int_0^t f\, dK^2_B$ for $\forall t \in \mathbb{R}^1_+, f \in L^2(\mathbb{R}^1_+)$.

We introduce fermionic creation and annihilation processes (Hudson and Parthasarathy, 1986)

$$K^2_F(t) = \int_0^t R\, dK^2_B, \qquad K^3_F(t) = \int_0^t R\, dK^3_B \tag{2.0$'$}$$

where $R = (-I)^{K_1}$ is the parity process, so that $dK^2_F = R\, dK^2_B$, $dK^3_F = R\, dK^3_B$. Since $R^2 = 1$, $dK^2_B = R\, dK^2_F$, $dK^3_B = R\, dK^3_F$, and the fermion field operators $b(f)$, $b^\dagger(f)$ are

$$b(f) = \int \bar{f}\, dK^3_F = \int \bar{f}R\, dK^3_B, \qquad b^\dagger(f) = \int f\, dK^2_F = \int fR\, dK^2_B$$

The operators $b^\#(f)$ are bounded and satisfy the CAR $[b(f), b(g)]_+ = [b^\dagger(f), b^\dagger(g)]_+ = 0$, $[b(f), b^\dagger(g)]_+ = \langle f, g\rangle 1, f, g \in L^2(\mathbb{R}^1_+)$, and, as the vacuum is annihilated by the $b(f)$ and is cyclic for the $b^\dagger(f)$, provide the Fock representation of the CAR.

From now on we shall omit the index F of K^i_F, $i = \overline{2, 3}$, and consider dK^i, $i = \overline{2, 3}$, of (2.0)$'$ and K^1 of (2.0) as basic martingales of fermionic quantum stochastic calculus. The Itô multiplication rules of the basic differentials dK^i, $i = \overline{1, 3}$, and the time differential dt, denoted here by dK^4 (Hudson and Parthasarathy, 1986; Parthasarathy, 1992), provides the identities

$$dK^1dK^1 = dK^1, \qquad dK^1dK^2 = dK^2, \qquad dK^3dK^1 = dK^3, \qquad dK^3dK^2 = dK^4$$

and all the others are equal to zero.

Given the adapted process E_i, $i = \overline{1, 4}$, the stochastic integral $M(t) = \int_0^t E_i\, dK^i$ (repeated suffix summation convention) is adapted, and so is its adjoint $M^\dagger(t) = \int_0^t E_i^\dagger\, dK^i$. The Itô type stochastic integrals are constructed and studied in Hudson and Parthasarathy (1984) and Parthasarathy (1992).

We identify the processes K^i, $i = \overline{2, 3}$, with their $\mathbb{Z}_2$-graded ampliations to the domain $h_0 \underline{\otimes} \mathscr{E}$, so that each $1_{h_0} \underline{\otimes} K^i$, $i = \overline{2, 3}$, acts on $u \otimes \psi(f)$ by the rule $1 \hat{\underline{\otimes}} K^i(t)(u \otimes \psi(f)) = R_0 u \otimes K^i \psi(f)$.

Let the process $U(t)$ be a solution of the fermionic q.s.d.e.

$$dU = Ul_i dK^i, \qquad U_0 = 1 \tag{2.1}$$

where the coefficients l_i, $i = \overline{1, 4}$, are a bounded operator in h_0 identified with their $\mathbb{Z}_2$-graded ampliations to $h_0 \otimes \Gamma_+$, satisfying the certain unitarity condition [originally see Hudson and Parthasarathy (1984); in the fermion case, Hudson (1993) and revised in Hudson and Struleckaja (1995)].

Let $U_{s,t} = U_s^{-1} U_t$, $0 < s \le t < \infty$, be a two-parameter family of operators on $h_0 \otimes \Gamma_+ = h_0 \otimes h_t \otimes h^t$. Then, for $0 < r \le s \le t < \infty$, we have

$$U_{r,s} U_{s,t} = U_{r,t} \tag{2.2}$$

We regard (2.2) as describing a two-parameter evolution. The family $\{U_{s,t}: 0 < s \le t < \infty\}$ is called the *stochastic evolution* associated with the unitary process $t \mapsto U_t$ on $h_0 \otimes \Gamma_+$, that is the solution of the q.s.d.e. (2.1).

Definition [Compare with Hudson and Lindsay (1987)]. A *fermionic cocycle* on $h_0 \otimes \Gamma_+$ is a family $\{U_{s,t}: 0 < s \le t < \infty\}$ of even unitary operators satisfying the following conditions for all $0 < r \le s \le t < \infty$:

$$\left.\begin{array}{rl} \text{(i)} & U_{r,s} U_{s,t} = U_{r,t} \\ \text{(ii)} & U_{s,t} \text{ is an even element of von Neumann algebra } \mathcal{N}_{s,t} \\ \text{(iii)} & \Gamma(S_s)^* U_{s,t} \Gamma(S_S) = U_{0,t-s} \\ \text{(iv)} & t \mapsto \mathrm{E}_{0,s}[U_{s,t}] \text{ is norm continuous} \end{array}\right\} \tag{2.3}$$

(We say that $U_{s,t}$ is even if it belongs to a subalgebra $\mathcal{N}_I^+$.) We observe that only uniform continuous semigroups are of interest to us. The following holds.

Theorem 1. If U is a solution of the q.s.d.e. (2.1), with $l_i \in \mathcal{N}_0$, $i = \overline{1, 4}$, constant, and

$$l_1 = w - I, \qquad l_2 = l, \qquad l_3 = -l^* w^\gamma, \qquad l_4 = ih - \tfrac{1}{2} l^* l \tag{2.4}$$

for unitary $w^* = w^{-1}$, h self-adjoint, l arbitrary in $B(h_0)$, and γ being a parity automorphism, then $\{U_{s,t} = U(s)^* U(t);\ 0 \le s \le t\}$ is *a fermionic cocycle.*

Sketch of the Proof. The uniqueness of the solution of the q.s.d.e. (2.1), which is unitary valued, but with an arbitrary initial value, and its coefficients satisfy the conditions (2.4) and fermion unitarity conditions (Hudson and Struleckaja, 1995), is obtained by the straightforward generalization to the $\mathbb{Z}_2$-graded case of Theorem 7.1 by Hudson and Parthasarathy (1984) (in the boson case) and proven with the initial value 1 in the fermion case in Theorem 6.2 by Applebaum (1987). From this it follows that (2.3i) is fulfilled.

Denote $U_s(x) = U_s^{-1}U_{s+x}$ for $U_{s,s+x}$, while fixing $s \in \mathbb{R}_+^1$. Now, $U_s(x)$ satisfies the integral equation

$$U_s(x) = I + \int_0^x U_s(\tau)l_i \, dK_s^i(\tau) \tag{2.5}$$

where $K_s^i(\cdot) := K^i(s + \cdot) - K^i(s)$, $i = \overline{1, 3}$, are new basic martingales. Then the Picard iteration procedure can be applied (Hudson and Parthasarathy, 1984), and the (strong) limit belongs to an algebra $\mathcal{N}_{s,t+s}^+$ as long as each of the iterates belongs to this algebra. This gives (2.3ii).

By arguments similar to ones in Hudson and Lindsay (1987), we show that (2.3iii) holds. Indeed, we observe that $\Gamma(S_s)^*\mathcal{N}_{s,t}\Gamma(S_s) = \mathcal{N}_{0,t-s}$. Let $V(\tau)$ denote an adapted process $\tau \to \Gamma(S_s)^*U(s)U(s + \tau)\Gamma(S_s)$. Then applying the first fundamental formula of quantum stochastic calculus (Hudson and Parthasarathy, 1984; Parthasarathy, 1992) we have for $f, g \in L^2(\mathbb{R}_+^1)$, $u, v \in h_0$,

$$\begin{aligned}
&\left\langle u \otimes \psi(f), \Gamma(S_s)^* \int_s^{s+t} U(s)^*U(x)l_i \, dK^i(x) \, \Gamma(S_s)v \otimes \psi(g) \right\rangle \\
&= \int_s^{s+t} [\langle u \otimes \psi((S_sf)(x)), U(s)^*U(x) \\
&\quad \times \overline{((S_sf)(x))}((S_sg)(x))l_1 v \otimes \psi(S_sg)(x))\rangle \\
&\quad + \langle u^\gamma \otimes \psi((S_s(-f\chi_{[0,x]}))(x)), U(s)^*U(x) \\
&\quad \times \overline{(S_s(-f\chi_{[0,t]})(x))}l_2^\gamma v \otimes \psi((S_sg)(x))\rangle \\
&\quad + \langle u \otimes \psi((S_sf)(x)), U(s)^*U(x)((S_s\,(-g\chi_{[0,t]}))(x)) \\
&\quad \times l_3R_0v \otimes \psi((S_s(-g\chi_{[0,x]}))(x))\rangle \\
&\quad + \langle u \otimes \psi((S_sf)(x)), U(s)^*U(x)l_4v \otimes \psi((S_sg)(x))\rangle] \, dx \\
&= \left\langle u \otimes \psi(f), \int_0^t \Gamma(S_s)^*U(s)^*U(s + \tau)\Gamma(S_s)l_i \, dK^i(\tau) \; v \otimes \psi(g) \right\rangle
\end{aligned}$$

Notice that the second quantization $\Gamma(S_s)$ commutes with the parity operator $\Gamma(-I)$ over the whole Fock space, and so is even. Thus, V also satisfies (2.5). By the uniqueness arguments, $V = U$. Finally, we observe that $\|P(t) - P(s)\| \le \|l_4\|(t - s)$, $0 \le s \le t < \infty$. ∎

Thus, this proves that the correspondence between the cocycles and unitary evolution in our case is closely parallel to the bosonic case (Hudson and Lindsay, 1987).

3. BACK TO UNITARY EVOLUTIONS

We complete this paper by stating the converse problem.

Theorem 2. Let $U_{s,t}$ be a *fermionic cocycle*. Then there exist $l_i \in \mathcal{N}_0$, with l_1, l_4 even l_2, l_3 odd, as in (2.4) and satisfying certain unitarity conditions, such that $U(t) \equiv U_{0,t}$ is the solution of the fermionic q.s.d.e.

$$dU = Ul_i\, dK^i, \qquad U_0 = 1, \quad i = \overline{1, 4}$$

The proof will appear in Struleckaja (1994).

Remark. Once again, we would like to emphasize that the 'unification' technique and the use of the 'exponential-vectors-span' formulation of the theory used in this paper, rather than the 'multiparticle-vector' formulation (Applebaum, 1987), enabled us to proceed with the generalization in a rather natural way.

ACKNOWLEDGMENT

I wish to express my gratitude to R. L. Hudson for stating the problem, and for his encouragement and many helpful discussions.

REFERENCES

Accardi, L., and Mohari, A. (1994). Preprint, University of Rome II, Rome, Italy.

Accardi, L., Journé, J.-L., and Lindsay, J. M. (1987). In *Quantum Probability and Related Topics IV*, W. v. Waldenfels and L. Accardi, eds., World Scientific, Singapore.

Applebaum, D. B. (1987). *Publications RIMS, Kyoto University*, **23**(1), 17–56.

Chevalley, C. (1955). *Publications of the Mathematical Society of Japan I*, Princeton University Press, Princeton, New Jersey.

Cockroft, A., and Hudson, R. L. (1978). *Journal of Multivariant Analysis*, **7**, 107.

Davies, E. B. (1971). *Transactions of the American Mathematical Society*, **158**, 115.

Fagnola, F. (1991). Preprint, University of Trento, Trento, Italy.

Hudson, R. L. (1993). *International Journal of Theoretical Physics*, **32**, 2413–2421.

Hudson, R. L., and Lindsay, J. M. (1987). *Mathematical Proceedings of the Cambridge Philosophical Society*, **102**, 363.

Hudson, R. L., and Parthasarathy, K. R. (1984). *Acta Applicandae Mathematicae*, **2**, 353.

Hudson, R. L., and Parthasarathy, K. R. (1986). *Communications in Mathematical Physics*, **104**, 457.

Hudson, R. L., and Shepperson, P. A. (1992). In *Proceedings, Quantum Probability and Related Topics*, VII, L. Accardi, ed., World Scientific, Singapore.

Hudson, R. L., and Struleckaja, V. R. (1995). *International Journal of Theoretical Physics*, to appear.
Journé, J.-L. (1987). *Probability Theory and Related Fields*, **75**, 291.
Parthasarathy, K. R. (1992). *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel.
Parthasarathy, K. R., and Sinha, K. B. (1986). *Journal of Functional Analysis*, **67**, 126.
Struleckaja, V. R. (1994). Preprint, University of Nottingham, to appear.